On Consistency / Handling Inconsistency

Paraconsistent Logic - Many-Valued Logic

*** Supplementary Slides ***

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After the lectures the students can...

1. Explain the logical concepts related to consistency and paraconsistency
2. Use a many-valued logic for inconsistency handling in simple situations
3. Provide a brief overview of approaches to inconsistency handling in AI

For the last item these links to the Stanford Encyclopedia of Philosophy are very relevant:

plato.stanford.edu/entries/logic-paraconsistent/
plato.stanford.edu/entries/logic-manyvalued/
We briefly discuss some further results using the necessity operator $\square$ which is an abbreviation $\square \varphi \equiv \varphi \iff \top$ but can alternatively be defined as follows:

$\llbracket \square \varphi \rrbracket = \begin{cases} 
\bullet & \text{if } \llbracket \varphi \rrbracket = \bullet \\
\circ & \text{otherwise}
\end{cases}$
#123 Ann believes that
if Joe studies then he gets good grades

#456 Ann believes that
if Joe does not study then he enjoys college

#789 Ann believes that
if Joe does not get good grades then he does not enjoy college

#999 Ann believes that Joe does not get good grades

If we assume that #999 is better captured by □¬G rather than simply ¬G then we still have that neither G, S, E, B or ¬B follows (the previous counter-examples in these cases all give ¬G the truth value • so the necessity operator □ makes no difference in these particular cases)

And of course ¬G still follows
However, as we shall see, \( \neg S \land \neg E \) now follows (and \( \neg S \) and \( \neg E \) follow separately as well)

This is quite acceptable; remember that in classical logic we would have explosion and we are now forcefully asserting (using the necessity operator \( \Box \)) the classical contradiction \( \neg G \) (with respect to the other assertions) so some implications are to be expected

And we even shall see that \( S \lor E \) follows (but as mentioned neither \( S \) nor \( E \) follows separately)

At first glance this may look odd since \( S \lor E \equiv \neg(\neg S \land \neg E) \), but such things are no problem in a paraconsistent logic
The consequences just mentioned can be proved as follows

If we are going to have a counter-example where \( \neg G \) is • then we can assume that \( \neg E \) is • or 1 (it could be 11 but that case is symmetric to the 1 case)

We have that assumption due to \( \neg G \rightarrow \neg E \)

And then from \( S \rightarrow G \) we have that \( S \) is o or 1, hence \( \neg S \) is • or 1

If both \( \neg S \) and \( \neg E \) are • then we are done with the main implication (for \( \neg S \land \neg E \) or \( S \lor E \) as right hand side of the main implication); and if at least one of them is 1 then the left hand side of the main implication is 1 and the right hand side is 1 too, and we are done
We consider a small knowledge base in the domain of medicine originally investigated by N. C. A. da Costa and V. S. Subrahmanian.

Three experts in medicine provided information related to the diagnosis of two diseases: disease-1 and disease-2.

The information concerning John and Mary can be paraphrased as follows...
Case Study II

— Expert I (a clinician):

*Symptom-1 and symptom-2 together imply disease-1.*
*Symptom-1 and symptom-3 together imply disease-2.*
*Disease-1 and disease-2 exclude each other.*

— Expert II (also a clinician):

*Symptom-1 and symptom-4 together imply disease-1.*
*Symptom-3 implies disease-2 if symptom-1 is not present.*

— Expert III (a pathologist):

*Only John has symptom-1 and symptom-4.*
*Neither John nor Mary have symptom-2.*
*Both John and Mary have symptom-3.*
Case Study III

Clearly the above information is classically inconsistent, since John both has and doesn’t have disease-1 and disease-2.

Hence from a straightforward formalization in classical logic we would also infer that Mary both has and doesn’t have disease-1 and disease-2, but the sensible result would be to infer just that Mary has disease-1 and doesn’t have disease-2 since the inconsistency with respect to John should not lead to inconsistency with respect to Mary.

Of course we could separate the information about John and Mary completely (in two separate knowledge bases), but we would still be able to infer that John has, say, some other disease-3 (and doesn’t have disease-3).

It would be preferable to remove the inconsistency, but that might not be possible, either for theoretical reasons — what are the principles to be used in order to revise the knowledge base? — or for practical reasons — how can hundreds or thousands of evolving rules be kept consistent?
Case Study IV

We use the abbreviations:

\[ \varphi \triangleright \psi \equiv \varphi \rightarrow \neg \psi \quad \varphi \triangleright \equiv (\varphi \triangleright \psi) \wedge (\psi \triangleright \varphi) \]

We could also have used \( \Box \neg \varphi \lor \Box \neg \psi \lor (\varphi = \psi \wedge \neg \Box \varphi) \) for \( \varphi \triangleright \equiv \).

We use the operator \( \triangleright \triangleright \) to express the “exclusion rule” of expert 1.
Case Study V

We use the logical necessity modality operator $\Box$ to express that the observations of expert III concerning symptoms are not — for the sake of simplicity — allowed to be inconsistent.

We also use the operator $\Box$ in the “exclusion rule” of expert I; we discuss some variants later.

A formalization is as follows with $D_i$ for disease-$i$, $S_i$ for symptom-$i$, $J$ for John and $M$ for Mary:

\[
S_1 x \land S_2 x \rightarrow D_1 x \quad S_1 x \land S_3 x \rightarrow D_2 x \quad \Box(D_1 x \leftrightarrow D_2 x)
\]

\[
S_1 x \land S_4 x \rightarrow D_1 x \quad \neg S_1 x \land S_3 x \rightarrow D_2 x
\]

\[
\Box S_1 J \quad \Box \neg S_2 J \quad \Box S_3 J \quad \Box S_4 J
\]

\[
\Box \neg S_1 M \quad \Box \neg S_2 M \quad \Box S_3 M \quad \Box \neg S_4 M
\]
We refer to these formulas as $\Pi$.

We now calculate the truth values for the knowledge base $\Pi$. In this case it is enough to consider four truth values since further truth values will interact in a similar manner.

We do the calculation by splitting $\Pi$ into $\Pi_J (x = J$ in $\Pi)$ and $\Pi_M (x = M$ in $\Pi)$ and using the truth tables we get the following two intermediate tables that must then be combined.
### Case Study VII

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Case Study VIII

The columns $\Pi_J'$ and $\Pi_M'$ correspond to the situation where $\Box$ is omitted from the “exclusion rule” for expert I.

From the columns $\Pi_J$ and $\Pi_M$ we obtain:

\[
\begin{align*}
\Pi & \vdash D_1 J & \Pi & \vdash \neg D_1 J & \Pi & \vdash \neg D_1 M & \Pi & \nvdash D_1 M \\
\Pi & \vdash D_2 J & \Pi & \vdash \neg D_2 J & \Pi & \vdash D_2 M & \Pi & \nvdash \neg D_2 M
\end{align*}
\]
We consider here just the details of the first result, namely $\Pi \models D_1 J$.

For the combination $\Pi$ we first observe that both columns $\Pi_J$ and $\Pi_M$ have $\top$ and $\bot$ rows, so $\Pi$ can be $\top$ and $\bot$ ($\Pi$ will never be $\bot$ since $\Pi_J$ is never $\bot$, cf. the truth table for the conjunction operator $\land$).

We then observe that when $\Pi$ is $\top$ then $D_1 J$ is $\top$ and using the truth table for the implication operator $\rightarrow$ we get $\bot$ (the designated truth value).

Similarly for $\bot$ and hence we have $\Pi \models D_1 J$. 
Case Study X

Given that the information is classically inconsistent we find these results the best possible because the inconsistency with respect to John does not lead to inconsistency with respect to Mary.

Some final comments...

From the columns $\Pi' J$ and $\Pi' M$ we see that if $\square$ is omitted from the “exclusion rule” we would not be able to derive $\neg D_1 J$, $\neg D_1 M$ or $\neg D_2 J$.

Instead of $\leftrightarrow$ we could consider Sheffer’s stroke $|$ (where $\varphi | \psi$ is equivalent to $\neg(\varphi \wedge \psi)$), but $\square(D_1 x | D_2 x)$ would not give any models — all rows are $\circ$ (the $\square$ is needed for the same reason as in the $\leftrightarrow$ case).

However, $\square(D_1 x \leftrightarrow D_2 x \lor D_1 x | D_2 x)$ is an interesting combination where we can derive $\neg D_1 J$ and $\neg D_2 J$, but not $\neg D_1 M$ since $D_1 M = \|$ and $D_2 M = \|=\|$ give $\|$ (and also $D_1 M = \|=\|$ and $D_2 M = \|\|$ give $\|$).