Remarks: In order to be permitted to the exam,
- an average of 50% of each exercise sheet has to be obtained,
- and on all but one exercise sheet at least 20% of the points have to be obtained,
- and exercises have to be solved (and submitted) in groups of at most 2 students,
- and everybody needs to show successfully twice in the exercise groups his/her solution (“an der Tafel vorrechnen”).

Exercise 1 (6 Points, Validity, truth, and satisfiability)
Decide for each of the following sentences whether it is (1) a tautology, (2) satisfiable but no tautology, or (3) unsatisfiable and prove it.
(a) \(\neg \text{top} \rightarrow \neg \text{small} \rightarrow \neg \text{top}\)
(b) \((\neg \neg p \rightarrow \neg r) \rightarrow (p \rightarrow r)\)
(c) \(\neg \text{red} \land (\text{red} \lor \text{green}) \land (\neg \neg \text{green} \rightarrow \text{red})\)
(d) \(\Box \rightarrow (\top \land (\neg \text{perfect} \lor \neg r \leftrightarrow (q \land s))\)

Exercise 2 (7 Points, Models)
Consider \(\text{Prop} =_{\text{def}} \{ r, s, t \} \). How many SL valuations (models) over \(\text{Prop} \) are there for the following formulae? For each formula, state an equivalent formula in the language of SL (i.e. no macros allowed) that is as short as possible (the number of symbols (not counting parentheses) is as small as possible). We are omitting parentheses for better readability when irrelevant for the truthvalue of the formula.
1. \(\neg (\neg r \rightarrow r)\)
2. \(\neg ((r \rightarrow \neg s) \land (t \rightarrow \neg s))\)
3. \(r \leftrightarrow t \leftrightarrow r\)
4. \(\neg (\Box \land ((r \land s) \rightarrow (s \lor t)))\)

Exercise 3 (13 Points, Entailment)
In the following, you are given two theories involving unicorns. Consider for each of the theories the questions: (1) Is the unicorn mythical? (2) Is the unicorn magical? (3) Is the unicorn horned? Prove which of these properties follow from the theory and which do not (i.e. are not entailed).
(a) There exists exactly one unicorn. If the unicorn is magical, then it is immortal, but if it is not magical, then it is a mortal mammal. If the unicorn is immortal or a mammal, then it is horned. The unicorn is mythical if it is horned.
(b) There exists exactly one unicorn. If the unicorn is magical, then it is immortal, but if it is not magical, then it is a mortal mammal. If the unicorn is either immortal or a mammal, then it is horned. The unicorn is mythical if it is horned.

To be submitted:
25. April 2016
in class
Exercise 4 (5 Points, Hilbert Calculus)

Use the Hilbert calculus from Definition 2.19 and show the following.

(a) \( \phi \lor \psi \vdash_{SL} \neg \neg \phi \lor \psi \),

(b) \( \neg \neg \phi \lor \psi \vdash_{SL} \phi \lor \psi \).

(c) Consider the theory \( T = \{ (c \rightarrow d), (a \rightarrow (b \rightarrow c)), (c \lor d), b, a \} \).
   
   1. Prove that \( T \vdash c \) by means of the Hilbert calculus from Definition 2.19.
   
   2. Prove that \( T \vDash c \). (It is not allowed to use the completeness result of the calculus!)

(Explain all steps!)

Exercise 5 (0 Points, Extra (15P): Boolean connectives)

In the lecture, we have defined SL with only the two connectives \( \neg \) and \( \lor \). All other connectives \( \land, \rightarrow, \leftrightarrow \) were defined as macros. We mentioned, that we could have also used \( \neg \) and \( \land \) as the only connectives.

In this exercise, we consider the question: Can we build SL on just one single binary Boolean connective \( \uparrow \)? So we define \( \text{Fml}_{SL(L\text{Prop})} \) (Definition 2.2) as follows

\[ \varphi ::= \square \mid p \mid (\varphi \uparrow \varphi) \]

where \( p \in \text{Prop}. \)

(a) How many semantics for a binary connective \( \uparrow \) do exist?

(b) How do they look like for these versions of SL? Write them down explicitly in the style of Definition 2.5 of the lecture.

(c) There are exactly two semantics, denoted by \( \uparrow_1 \) and \( \uparrow_2 \), such that all other boolean connectives \( \neg, \rightarrow, \leftrightarrow, \lor, \land \) can be defined as macros. Find them.

(d) Find a semantics for \( \uparrow \) where \( \neg \) can not be expressed and prove that it can’t.

(e) Are the inference rules \( \frac{\alpha \uparrow_1 (\beta \uparrow_1 \gamma)}{\alpha \uparrow_1 \beta} \) and \( \frac{\alpha \uparrow_1 \beta}{\beta} \) correct inference rules in a calculus for \( \uparrow_1 \) or \( \uparrow_2 \)? Prove or disprove.

(f) Consider any formula in \( \text{Fml}_{SL(L\text{Prop})} \) that uses only one propositional constant, i.e. \( (a \uparrow_1 a) \uparrow_1 (a \uparrow_1 (a \uparrow_1 a)) \). Find a simple algorithm to determine whether such a formula is a tautology and prove its correctness.