Logic and Verification
– Sheet 4: FOL + Verification –

Exercise 1 (6 Points, Formalization)
Translate the following sentences into predicate logic. Specify the used predicate and function symbols.

1. Every rock band has a guitar player.
2. If anything is green, then A is also green while C is not.
3. If I am on a diet, I don’t eat any food that contains sugar.
4. A is above C, D is on E and above F, and if something is on F, then so is G.
5. Only fast cars are allowed on some highways.
6. A formula is logically valid if it is true in every interpretation.

Exercise 2 (10 Points, Formalization)
The signature Σ is given as follows (with the intended meaning in parentheses):
- unary predicate problem(x) (“x is a problem”),
- binary predicate solution(x, y) (“y is a solution for x”),
- unary predicate human(x) (“x is a human”),
- binary predicate knows(x, y) (“x knows y”).

(a) Formalize the following in FOL and use only the given predicate symbols:
1. For every problem there is a solution.
2. There is no problem which is known by all humans.
3. Each human knows some problem for which she knows not a single solution.

(b) What is the intuitive meaning of the following formula:

\[ \forall x (\forall y ((\exists z \text{ solution}(z, y) \rightarrow \text{knows}(x, y)) \rightarrow \neg \text{human}(x))) \]

Exercise 3 (10 Points, FOL with equality)
Consider the theory \( T = \{ \varphi_1, \varphi_2, \varphi_3 \} \) in FOL with equality and a single binary predicate \( p(\cdot, \cdot) \).

\[ \varphi_1 = \forall x \forall y \forall z (p(x, y) \land p(y, z) \rightarrow p(x, z)) \]
\[ \varphi_2 = \forall x \forall y ((p(x, y) \land p(y, x)) \rightarrow x = y) \]
\[ \varphi_3 = \forall x \exists y (\neg (x = y) \land p(x, y)) \]

1. Is \( T \) satisfiable?
2. Is there a finite model of $T$?
3. Answer the same questions for the theories $\{\varphi_1, \varphi_2\}$, $\{\varphi_1, \varphi_3\}$, $\{\varphi_2, \varphi_3\}$.

Exercise 4 (8 Points, Formalizing in FOL)
Prove or disprove the following:
1. $(U, I) \models \neg \exists x ((\forall z Q(z)) \lor P(-1, f(x)))$ with
   - $U = \mathbb{Z}$ (integers),
   - $I(P) = \{(x, y) \mid x = y + 1\}$,
   - $I(-1) = -1$,
   - $I(Q) = \{x \mid x \text{ is odd}\}$ and
   - $I(f): \mathbb{Z} \to \mathbb{Z}$ with $I(f)(x) = x - 3$.
2. $(\forall x (P(x) \lor \varphi)) \leftrightarrow ((\forall x P(x)) \lor \varphi)$
3. $(\forall x (P(x) \to \varphi)) \leftrightarrow ((\exists x P(x)) \to \varphi)$
4. $(\forall x (P(x) \to \varphi)) \leftrightarrow ((\forall x P(x)) \to \varphi)$

Exercise 5 (7 Points, Hoare logic)
Consider the following program $\text{mag}(x)$:
\[
\begin{align*}
    z &:= 0; \\
    \text{while } x \neq 0 \{ \\
    &z := z + 5; \\
    &x := x - 1 \\
    \}
\end{align*}
\]
1. Which of the following statements are correct? Explain your answers.
   - (a) $\models^t \{ x > 0 \} \text{mag}\{z > 0\}$
   - (b) $\models^t \{ x > 0 \} \text{mag}\{z = 5x\}$
   - (c) $\models^p \{ x < 0 \} \text{mag}\{z = 7\}$
   - (d) $\models^p \{ x > 0 \} \text{mag}\{z > 0\}$
   - (e) $\models^p \{ \top \} \text{mag}\{z > 0\}$
   - (f) $\models^t \{ x = x_0 \land x > 0 \} \text{mag}\{z = 5x_0\}$
2. Change the program so that it is partially correct for postcondition $\{z = 5x\}$ and precondition $\{x < 0 \lor x \geq 0\}$

Exercise 6 (0 Points, Extra (10P): “Is any re set of axioms already recursive?”)
Consider a first-order theory $T$ that is defined by a recursively enumerable, but not recursive, axiom system.
Is it possible to construct a recursive axiom system for $T$? Prove or disprove.