Artificial Intelligence  
– Sheet 6: Sentential Logic and Hoare Calculus –

Exercise 1 (8 Points, Boolean connectives)

In the lecture, we have introduced the Boolean connectives \( C := \{ \neg, \lor, \land, \rightarrow \} \).

(a) (2 Points) Show that the Boolean connectives \( \{ \neg, \land \} \) are sufficient. That is, show that each connective from \( C \) can be defined from just these two.

(b) (3 Points) Consider the binary Boolean function \( | : \{0, 1\}^2 \to \{0, 1\} \) defined by:

\[
(1, 1) \mapsto 0, \quad (0, 1) \mapsto 1, \quad (1, 0) \mapsto 1, \quad (0, 0) \mapsto 1
\]

Can all Boolean connectives from \( C \) be defined by “|”? Prove or disprove!

(c) (3 Points) Suppose \( \phi \) contains only the constant \( a \) (perhaps at various occurrences) and \( | \) (e.g. \( a | (a | a) | a), a | (a | a) \)). Can you find a very simple algorithm to determine whether the formula is a tautology or not?

Exercise 2 (6 Points, Resolution)

(a) (2 points) Given the theory \( T = \{ A \rightarrow \neg B, \neg A \rightarrow (C \land D), B \} \). Use resolution calculus to show that \( T \models D \) holds. Explain your procedure!

(b) (2 points) Describe, using the resolution calculus, how to show that a formula \( \varphi \) is a tautology.

(c) (2 points) Apply the method described in (b) in order to decide whether the formula \( ((A \land B) \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C)) \) is a tautology.

Exercise 3 (12 Points, Partial correctness)

Consider the following programs and use the annotation calculus to prove that the program is partially correct regarding the given pre and postconditions (note, that also the proof obligations have to be verified).

(a) if \( (x > y) \) {
\[
\begin{align*}
& z := x; \\
& \text{else} \\
& z := y
\end{align*}
\]

Precondition \( \top \), postcondition \( z = \max\{x, y\} \).

(b) \( z := 0; \)
\[
\begin{align*}
& \text{while } x > 0 \{ \\
& \quad z := z + x; \\
& \quad x := x - 1
\}
\]

Precondition \( x = x_0 \land x \geq 0 \), postcondition \( z = \frac{x_0(x_0+1)}{2} \).
Exercise 4 (6 Points, Total correctness)

Consider the following program `Mult`:

```plaintext
a := 0;
z := 0;
while (a \neq y) {
    z := z + x;
a := a + 1
}
```

Use the annotation calculus to prove that $\vdash^t \{y \geq 0\} Mult \{z = xy\}$